"Dynamic visual proofs" using DGS

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Abstract

Pictures and diagrams play an important role in the process of understanding various mathematical features. Moreover, an appropriate picture or diagram can be used as a visual proof of some geometric property or theorem. These non-verbal proofs, used to be called 'proofs without words', are more attractive and acceptable to students than the classical proofs. The use of dynamic geometry or algebraic software removes the weak point of these proofs which lie in the fact that they mostly do not capture the chain of thought leading to the proof but only the result. The paper presents selected examples of such dynamic visual proofs created by dynamic geometry software in the form of materials that the authors use in the teaching of secondary school mathematics and in the mathematics teacher training programs. This paper also provides a comparison to their related classical proofs from contemporary textbooks.

1 Introduction

Operations of reasoning, verification and proving are principal components of mathematics. They play by definition an irreplaceable role not only in mathematics building but also in its interpretation and understanding. No doubt reasoning and proving have their place in the mathematics curricula. Let us mention for example the Czech [10] or the United States [11] curricular documents. The reasoning and proving are not meant to be special topics in school mathematics but these techniques with the skills and knowledge corresponding to them are expected to pervade the whole mathematics content in these documents. Particular ways of applying the techniques of reasoning and proving in mathematics education have been widely discussed throughout the community of educators and educational experts [9, 14, 16]. The educational impact of practice in reasoning and proving in schools no doubt goes beyond the sphere of mathematics. The awareness of the importance of verifying certain claims make pupils and students more cautious when coming into contact with various manipulations of numbers and so help them, for example, to responsibly handle their personal finances in the future [15].

Various methods of proving can be distinguished according to the techniques or means used within them. A significant contribution to the methods of proving in mathematics has been made by computers. For example, we have recently witnessed the growing importance of the automated theorem proving [2, 14, 16, 17]. The role of "proof" in mathematics education is far from beeing settled, which can be obvserved in authoritative collections like [21]. However, the *notion* of "proof" needs to be clarified for this paper. With respect to "automated theorem proving" in geometry [2, 17], proofs are formal, based on logic theories and conformant with standards of academic mathematics. Opposed to this, in the science of mathematics education (for pedagogical reasons to be discussed), the reasoning is developed in geometric models (e.g.the Euclidean Model) introducing the notion of "visual proof" [14, 16]. Beeing aware of the controversial discussions we shall adopt the latter notion for this paper (without quotation marks). The increasing improvement of the qualities of dynamic geometry systems allows to verify various statements or to produce dynamic visual proofs. The latter use of the dynamic geometry system, specifically of the open source software GeoGebra [22], is dealt with in this paper.

The paper presents examples of original dynamic visual proofs of selected theorems that are featured as the non-verbal proofs, which used to be called 'proofs without words'. The examples, created with GeoGebra by the first author, who works as a grammar school mathematics teacher, will be presented within particular educational situations that arise from her teaching practice. Each situation corresponds to a rather different role [5] that 'proofs without words' can play in grammar school mathematics. Pros and cons of the use of 'proofs without words' in mathematics teaching, inferred from their utilization in secondary school mathematics and in the mathematics teacher training programs, will be discussed within the paper.

Dynamic visual proofs and their application in mathematics education are the subject of the doctoral research of the first author. Her focus on proofs was, among others, motivated by her personal experience. As a grammar school teacher she frequently faces students' lack of interest in proofs, often associated with the fear of failure. In her opinion this negative students' attitude towards proofs could be changed by the use of dynamic visual proofs. The paper serves to share the authors' first experience in this direction.

2 Proofs without words

Pictures and diagrams play an important role in the process of understanding various mathematical features. Moreover, an appropriate picture or diagram can be used as a visual proof of some geometric property or theorem. These non-verbal proofs, called 'proofs without words', are, as the authors have experienced, more attractive and acceptable to students than the classical algebraic proofs. Their educational potential is no doubt high, but compared to classical proofs, their static pictures do not capture the chain of thought leading to the proof but only the result. This deficiency, arising especially in the educational process, can be eliminated by the use of dynamic geometry software.

2.1 History

No wonder that the 'proofs without words' can be traced back to the remote past considering the leading role of geometrical argumentation in the development of the phenomenon of proving in mathematics. In Fig. 1 we can see the 6th century BC Greek drachma from Aegina island with the geometrical proof of the equality $(a + b)^2 = a^2 + 2ab + b^2$ without words on its reverse side, a clear example of linking mathematics with everyday life. The same picture was later, around 300 BC, used by Euclid in his Elements [7] followed by the strictly geometrical interpretation: "If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments."

Figure 1: Drachma of Aegina, 6th century BC [6]

2.2 Present

Nowadays many 'proofs without words' of various theorems and qualities are well documented in a number of publications. The most renowned among them are books [12, 13] by Roger Nelsen. These are a collection of excellent proofs from various areas of mathematics. Unfortunately their only disadvantage is inherently unavoidable. As with all printed material they suffer from statism.

Dynamic realization of 'proofs without words' can be found on various specialized web pages. First of all, let us mention [23] with hundreds of visual proofs. Many problems from different branches of mathematics, some of them followed by dynamic visual proofs are offered on [18]. Also [8] provides an interested person with miscellaneous dynamic visual proofs.

3 Proofs at grammar schools

As mentioned above reasoning and proving have an irreplaceable role in mathematics education. Apart from the fact that the techniques of reasoning and proving form mathematical science they also serve as the means of conveying mathematical knowledge and of practicing methods of logical reasoning. This role of reasoning and proving in mathematics education is fully accepted by teachers, educational experts and the government. In the official curriculum for Czech grammar schools (There are two main kinds of grammar school in the Czech Republic: four-year grammar school (age 15-19) and eight-year grammar school (age 11-19)) [10] the following expected outcomes are given in the section 'Logic and proof': "The pupil shall read and record theses in the symbolic language of mathematics, properly use logical connectives and quantifiers, distinguish between definitions and theorems, discern premises and conclusions of the theorems, discern between correct and incorrect judgments, create hypotheses, justify their validity or invalidity, and refute invalid statements, justify his/her approach and verify the validity of his/her steps in the solution of the problem."

Despite these facts and requirements the authors have experienced that secondary school students often do not like mathematical proofs. A significant number of students consider them difficult and useless and they would rather believe mathematical theorems than prove them, without any interest in their genesis. On the one hand, it is gratifying that students trust their teacher but on the other hand, a reasonable degree of doubt is the driving force of knowledge.

To find all the reasons for so little interest in mathematical proofs among secondary school students is a complex task. One of these reasons could be the textbooks that are used at secondary schools. They, of course, contain only static figures and the algebraic proofs prevail, even for the geometric theorems. No officially created dynamic materials on CDs or from the Internet are attached to textbooks. Other than textbooks we could give more reasons; for example the personality of a teacher, use of computers etc., but whatever we mention will be connected to motivation as a key factor of students' interest.

A very fresh experience shows that a new nationwide conception of the secondary school final exam, which used to be called 'maturita', that has been used since 2011 in the Czech Republic [20] could play a positive role in students' interest in mathematics. As a part of 'maturita' students have to choose between mathematics and a foreign language. This necessity of choice initiated an unexpected interest in mathematics by students. Moreover, establishing mathematics as a compulsory subject of 'maturita' is still being discussed. Most mathematics teachers agree that the maturita test tasks have been well formulated. A reasonable number of tasks require a deeper insight into the corresponding mathematical relations. Students suddenly find that understanding the subject matter is more useful than to learn it by heart. The space for useful and acceptable proofs is opened.

4 Selected examples

Proofs can perform various functions in mathematics and also in mathematics education. For example, Hanna states the following functions of proof in [5]: verification, explanation, conviction, systematization, discovery, communication and enjoyment. Examples that will be presented in this section will be settled in three particular educational situations, each focused on slightly different functions of dynamic visual proofs in mathematics teaching.

All presented examples are from geometry. The main reason for this is not the evident closeness of visual proofs to geometry but the current trend towards the reduction of the proportion of geometry taught in mathematics lessons in schools. This situation calls for the use of dynamic geometry software. Its proper utilization can significantly speed the teaching of geometry. Moreover it brings a new quality to the tuition in that students are more active and involved in the learning process.

4.1 Discovery: Area of trapezoid

This section deals with the use of dynamic materials - 'proofs without words' by students to derive a formula for the area of a trapezoid themselves. The presented situation has a place in the 2nd year (age 16) of mathematics in the four-year grammar school. In this situation the dynamic visual proof relates especially to the discovery function.

The students have already learnt the formula for the area of a triangle. Formulas for the area of some other polygons are given in their textbooks. Now they should learn how to derive and prove these formulas. The textbook instructs them to derive and prove the formulas for parallelogram, triangle and trapezoid using their transformations into one or more rectangles. They find the following assignment in their textbook [4].

Task assignment No. 1: *Use the given picture (see fig. 2) to derive a formula for the area of each of the given polygons.*

The textbook offers students picture, seen in fig. 2, to facilitate this task for them.

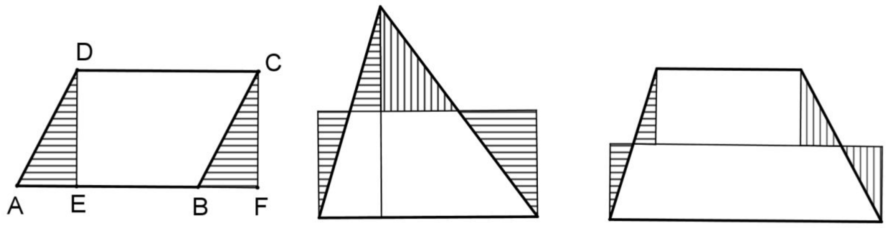

Figure 2: Derive formulas for the areas of given polygons,[4]

As mentioned we will trace the students' approach to the problem of the trapezoid. The authors have recognized that most students understand the meaning of the 'proof without words' in the picture only if they confront it directly with the formula, which is published in the textbook or students remember it from their previous school attendance. It is too complicated for most of them to use the picture as a visual aid for either remembering or deriving the formula. Many students prefer to learn it by heart rather than make the effort to derive it from a static picture. However, such an attitude lacking any geometrical anchor could someday backfire. For example facing a problem in a maturita test, as illustrated by the following example from its 2011 variant [20]:

What percent of the volume of the trapezoid ABCD (see in Fig. 3) is created by the volume of the triangle ACD?

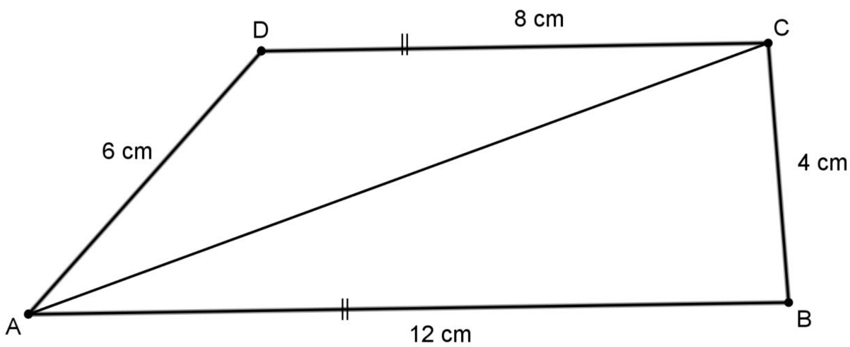

Figure 3: Maturita 2011, Problem No. 14, [20]

As opposed to the textbook the authors have found that it is better to start the formula derivation with a material that presents trapezoid as a composition of two triangles, as in fig. 3 from the quoted maturita task. The dynamic variant of the picture, seen in fig. 4, 5, that gradually reveals the trapezoid as a union of two triangles, has proven to be more effective in clarifying the facts and in addition is more attractive to students.

Task assignment No. 2: *You already know the formula for the area of a triangle. Use the given material (see fig. 4, 5) to derive a formula for the area of a trapezoid. Play with the material yourself and write the formula for the given lengths.*

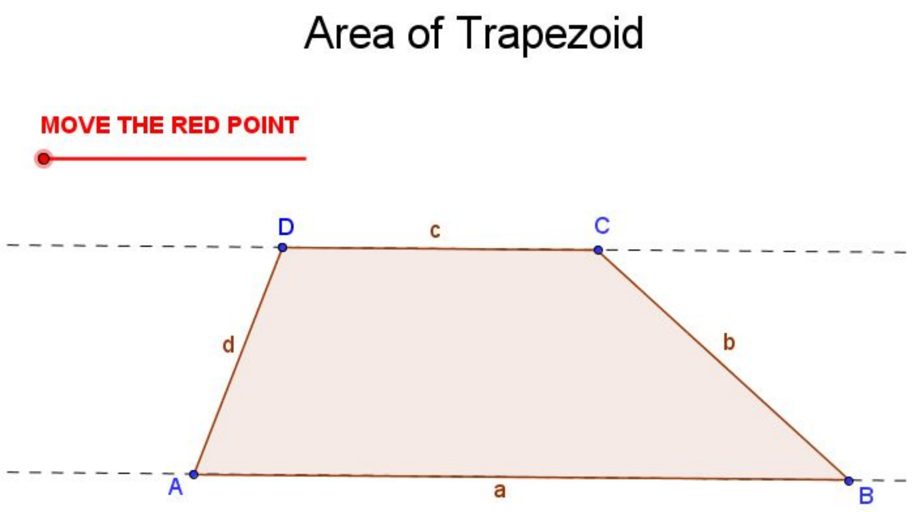

Figure 4: Area of trapezoid – division into two triangles (introductory phase) [25]. Dynamic visual proof created with GeoGebra [22]

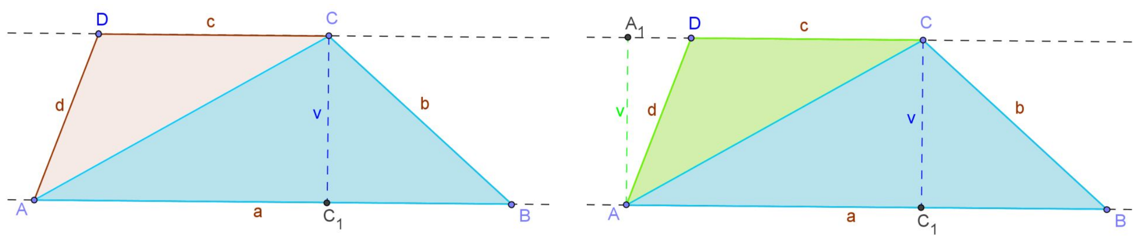

Figure 5: Area of trapezoid – division into two triangles (following phases) [25]. Dynamic visual proof created with GeoGebra [22]

The solving process of the task can be performed either individually in a computer lab or collectively by using a smart board. Students manipulate the dynamic picture and are soon able to derive (or justify the correctness of) the formula $S = \frac{1}{2}(a + c)v$ for the area of the trapezoid themselves. Then the visual proof in figure 2 can be introduced as another way of the formula derivation which uses the concept of mid-segment. The corresponding dynamic proof presented in fig. 6, 7 works according to a slightly different idea. It shows transformation of the trapezoid into one rectangle instead of two and directly uses the mid-segment length.

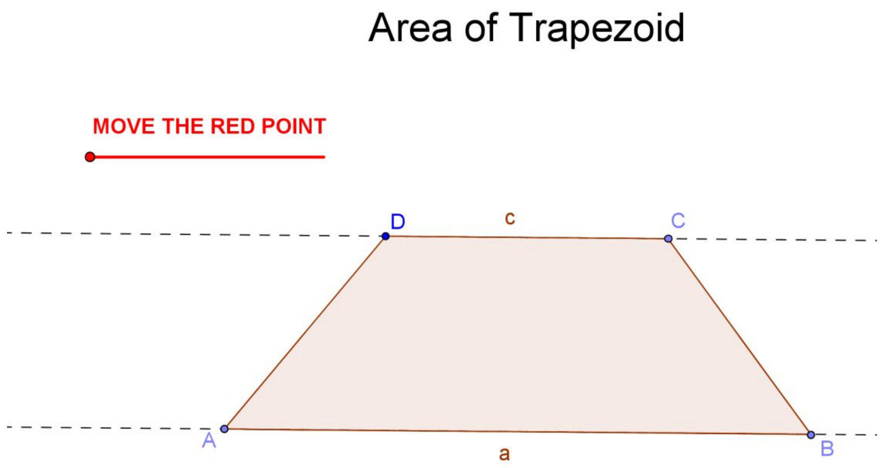

Figure 6: Area of trapezoid – use of mid-segment (introductory phase) [24]. Dynamic visual proof created with GeoGebra [22]

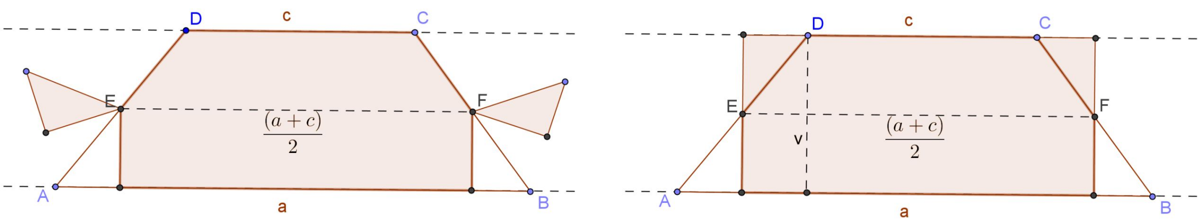

Figure 7: Area of trapezoid – use of mid-segment [24]. Dynamic visual proof created with GeoGebra [22]

4.2 Conviction: Law of sines

The dynamic visual 'proofs without words' served mainly as a means of given formulas derivation in the previous situation. They played the part predominantly assigned to dynamic figures, that are regarded mainly as a means of setting up conjectures and their preliminary verifications, rather than as a means of proofs. Now, we will present the use of the dynamic visual proof as a proof that really validates given theorem and provides a student with a credible conviction of its verity. Concretely we will deal with the 'Law of Sines'. This topic has a place in the 2nd year (age 16) of mathematics in the four-year grammar school. The law is stated in the following form in [3]:

For any triangle, where a, b, and c are the lengths of the sides of a triangle, α, β and γ are the opposite angles and R is the radius of the circumcircle

$$\frac{a}{sin\alpha} = \frac{b}{sin\beta} = \frac{c}{sin\gamma} = 2R.$$

The proof presented in the textbook [3] is rather vast, occupying over two pages. It is based on the definition of the sines, separately devided into acute angle, right angle and obtuse angle and connection with the circumscribed circle is not mentioned at all.

Instead of the textbook proof, the authors have worked to start with the dynamic visual proof that is based on the static geometrical proof of the law of sines published in [1] . This elegant visual proof, seen in fig. 8, is based on the features of the angles at the circumference, particularly the equality of the angles standing on the same arc and that an angle inscribed in a semicircle is a right-angle.

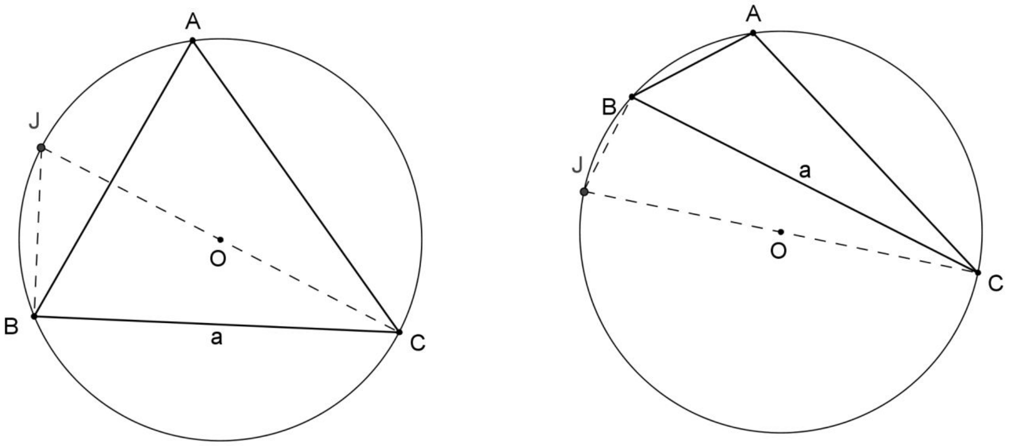

Figure 8: Law of sines. Static proof in [1]

There are in fact two corresponding dynamic visual proofs; one for an acute angle [26] and the second for an obtuse angle [27]. Utilizing them, either on a computer or on a smart board, students are provided with an immediate verification of the theorem. Then it is appropriate to let students analyze the proof given in the textbook [3] for themselves.

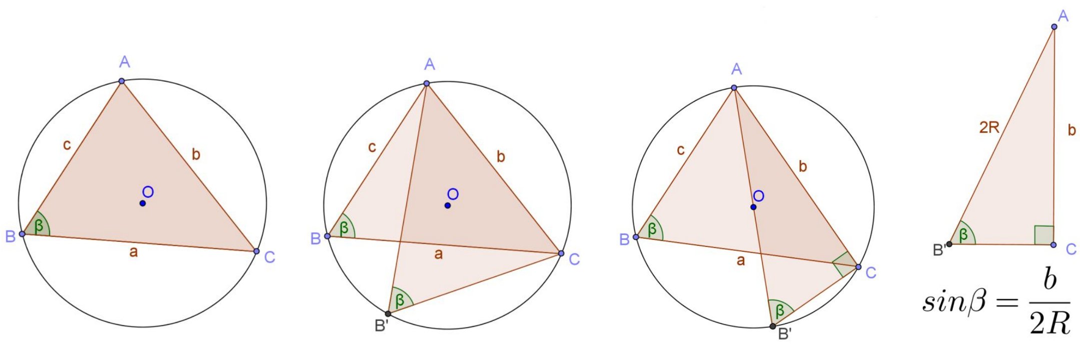

Figure 9: Law of sines – for acute angle [26]. Dynamic visual proof created with GeoGebra [22]

4.3 Verification: Right triangle altitude theorem

The paper does not advocate the general replacement of algebraic or geometric proofs with the 'proofs without words' in the teaching of mathematics. Its aim is to show that well prepared and properly used dynamic visual proof can perform various functions in various situations in the mathematics curriculum. Depending on the situation a dynamic visual proof can completely replace other types of proofs, facilitate their understanding or merely verify their correctness. In this section we will use the topic of the so called Euclid's theorems, which are stated together with the Pythagorean theorem in the Czech mathematics curriculum, to show the latter function of dynamic visual 'proofs without words'. Two of Euclid's theorems - 'of a leg' and 'of an altitude', which deal with the right-angled triangle, are given in [4] as predecessors of the Pythagoras' theorem in the following wording.

Euclid's theorem of leg: *For every right triangle square of the length of the leg is equal to the product of the hypotenuse and the segment of the hypotenuse adjacent to the leg.* [4]

The symbolic proof of the theorem is based on the similarity rules for triangles.

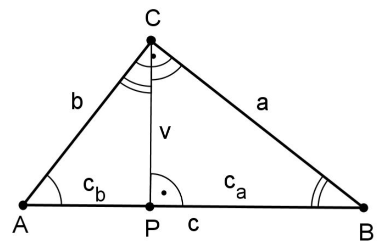

Figure 10: Euclid's theorems. Algebraic proofs in [4].

Let the right-angle triangle ABC be given with the right angle at C, see fig. 10. Then the partial triangles CBP and ABC are similar according the 'angle-angle similarity'. Then

$$a : c_a = c : a,$$

that can be simplified into the equation

$$a^2 = c \cdot c_a.$$

Equally, from the similarity of triangles ACP and ABC follows

$$b : c_b = c : b,$$

that gives the equation

$$b^2 = c \cdot c_b.$$

Which was what had to be proved.

The dynamic visual proof of the 'Euclid's law of a leg' in fig. 11 is without doubt striking but it can not replace the simple symbolic proof mentioned above. It verifies the theorem and provides another view of the problem.

Euclid's theorem of altitude: *For every right triangle square of the length of the altitude from the hypotenuse to the right angle is equal to the product of the two segments into which the hypotenuse is divided.*

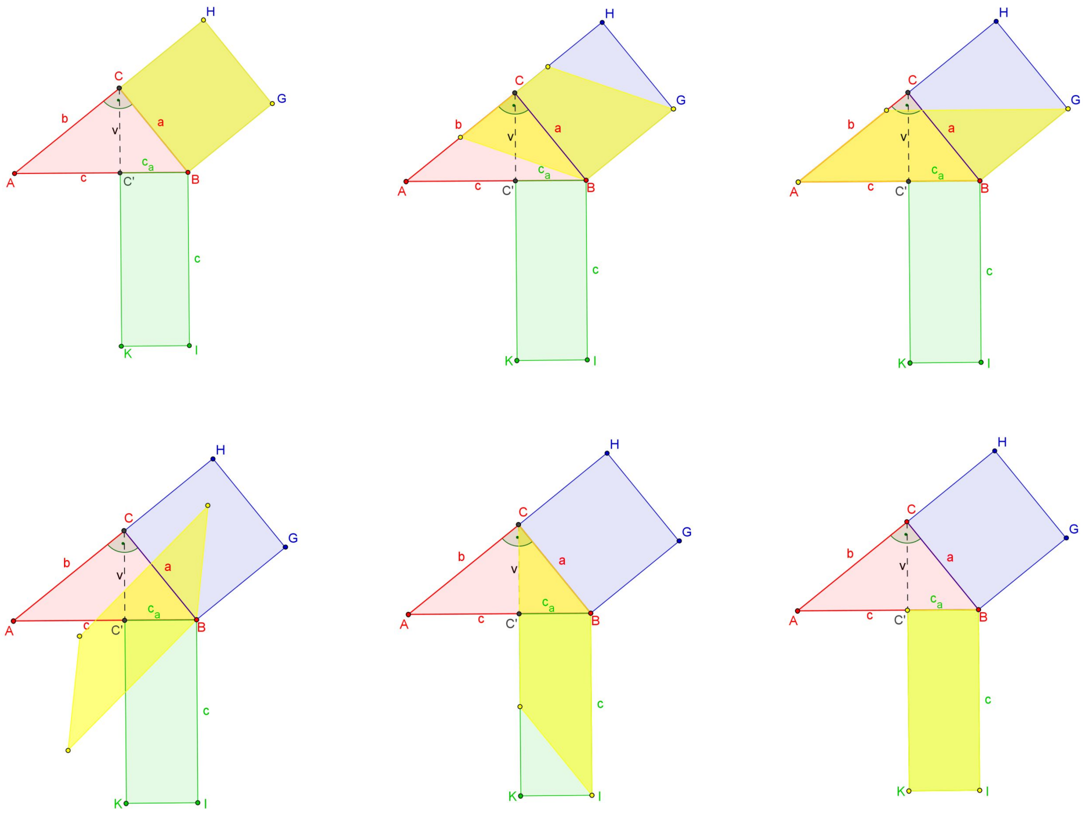

Figure 11: Euclid's theorem of leg [28]. Dynamic visual proof created with GeoGebra [22]

The proof of this theorem is an outstanding example of the situation where the symbolic proof is without doubt more effective than the dynamic visual proof. See fig. 10 again. The triangles ACP and CBP are similar. Therefore we can write the following equation

$$c_b : v = v : c_a,$$

that can be simplified into

$$v^2 = c_a \cdot c_b.$$

Which was what had to be proved. In contrast to its simplicity the geometric proof is not as clear and not too easy to understand, as you can see in fig. 12

5 Conclusion

In this article we showed three concrete examples of the use of dynamic geometrical 'proofs without words' in the teaching of mathematics, each in a slightly different role. We aimed to show that these dynamic materials have their place in mathematics education. As the authors have experienced, utilization of such materials requires some changes in the attitude of both, teachers and their students. From the students' point of view it is for example an active involvement in learning, argumentation

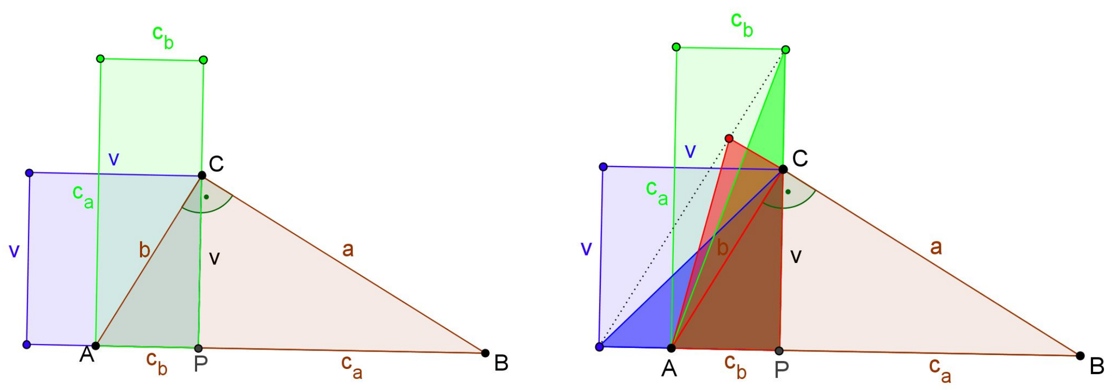

Figure 12: Euclid's theorem of altitude [29]. Dynamic visual proof created with GeoGebra [22]

skills to defend their solution or the ability to apply acquired knowledge in a wider context. Teachers should be able to choose appropriate topics, change the organization of teaching together with their methods of teaching and assessment, etc. These demands appear clear but to identify them all properly it is necessary to do a serious research in this direction. The first author takes advantage of her professional background and aims her doctoral research at this topic.

Acknowledgments

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Supplemental Electronic Materials

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